Quantum Morphisms

Lecture 5

Last Week

Introduced projective packing/clique numbers α_p / ω_p

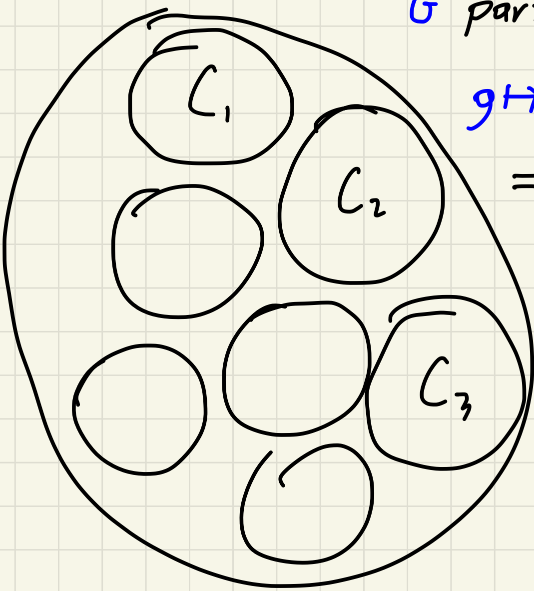
Relation to projective rank: $\xi_f(G) \geq \frac{|V(G)|}{\alpha_p(G)}$ & $\omega_p(G) \leq \xi_f(G)$

Introduced quantum clique/independence numbers ω_q / α_q

$\alpha(G) \leq \alpha_q(G) \leq \alpha_p(G) \leq \chi(\overline{G})$

If $\alpha_p(G) \doteq \chi(\overline{G})$, then $\alpha_q(G) = \chi(\overline{G})$.

G partitioned into $\chi(\overline{G})$ cliques $C_1, \ldots, C_{\chi(\overline{G})}$



$g \mapsto E_g$ proj. pack. of value $\chi(\overline{G})$

$\Rightarrow \sum_{g \in C_i} E_g = I \quad \forall i$

$F_{ig} = \begin{cases} E_g & \text{if } g \in C_i \\ 0 & \text{o.w.} \end{cases}$

gives quantum indpt set of size $\chi(\overline{G})$

Use Kochen-Specker sets to find graphs with $\alpha(G) < \alpha_q(G)$

Using $G \overset{q}{\nrightarrow} H$ & $G \nrightarrow H$ to get $\alpha(K) < \alpha_q(K)$

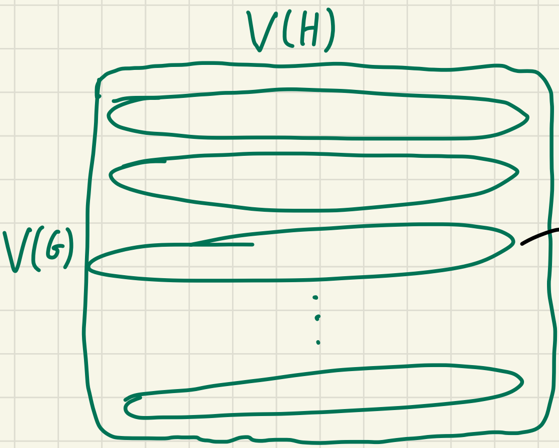
Homomorphic product $G \ltimes H$

$V(G \ltimes H) = V(G) \times V(H)$ $\qquad E_{gh} E_{g'h'} = 0$

$(g,h) \sim (g',h')$ if $(g = g'$ & $h \neq h')$ or $(g \sim g'$ & $h \nsim h')$

Special cases: $G \ltimes K_n = G \,\square\, K_n$ Cartesian product $(=,\sim)$ or $(\sim,=)$

$K_n \ltimes H = \overline{K_n \times H}$ Categorical product $(\sim,\sim)$



$\{(g,h) : h \in V(H)\}$ is a clique $\Rightarrow \chi(\overline{G \ltimes H}) \leq |V(G)|$

Lemma: $\alpha(G \ltimes H) \leq |V(G)|$ and equality holds if and only if $G \to H$. $\quad f$ hom. $\{(g, f(g)) : g \in V(G)\}$

$|S| = |V(G)|$ indpt set $\qquad g \mapsto h$ s.t. $(g,h) \in S$

Note: $\alpha_q(G \ltimes H) \leq \alpha_p(G \ltimes H) \leq \chi(\overline{G \ltimes H}) \leq |V(G)|$.

Theorem (Mančinska + me): The following are equivalent

1) $G \xrightarrow{q} H$,
2) $\alpha_p(G \ltimes H) \doteq |V(G)|$ (or $\alpha_p(G \ltimes H) \dot{\geq} |V(G)|$),
3) $\alpha_q(G \ltimes H) = |V(G)|$ (or $\alpha_q(G \ltimes H) \geq |V(G)|$).

First let's show $G \xrightarrow{q} H \Rightarrow \alpha_q(G \ltimes H) = |V(G)|$ in terms of the games.

Suppose A + B have a strategy for the (G,H)-hom game. They will play the $(K_{V(G)}, \overline{G \ltimes H})$-hom game as follows: Upon receiving $g \in V(K_{V(G)}) = V(G)$, they act as if this is their input for the (G,H)-hom game and obtain output $h \in V(H)$. They then respond with $(g,h) \in V(G \ltimes H)$.

A + B get g, g' output (g,h) & (g',h') resp.

$g = g' \Rightarrow h = h' \Rightarrow (g,h) = (g',h')$

$g \neq g'$: 1) $g \sim g' \Rightarrow h \sim h' \Rightarrow (g,h) \not\sim (g',h')$ in $G \ltimes H$ ($\neq$)

2) $g \not\sim g' \Rightarrow (g,h) \not\sim (g',h')$ in $G \ltimes H$ ($\neq$)

$F_{g(g,h)} = E_{gh}$ $\quad F_{g'(g,h)} = 0$ if $g' \neq g$.

$= \chi(\overline{G})$

$\alpha_q(G \ltimes H) = |V(G)| \Rightarrow \alpha_p(G \ltimes H) \doteq |V(G)|$ is immediate.

$\Leftarrow$

Suppose that $(g,h) \mapsto E_{gh} \in \mathbb{C}^{d \times d}$ is a proj. pack. of value $|V(G)|$, thus $\sum_{g,h} rk(E_{gh}) = d|V(G)|$.

We want to show that the E_{gh} satisfy the conditions for $G \ltimes H$.

1) Orthogonality conditions hold by definition of $G \ltimes H$.

2) Need to show $\sum_h E_{gh} = I \quad \forall g \in V(G)$.

By orthogonality $\sum_h E_{gh}$ is a projection $\forall g \in V(G)$, thus

$$d|V(G)| = \sum_{g,h} rk(E_{gh}) = \sum_g \left(\sum_h rk(E_{gh})\right) = \sum_g rk\left(\sum_h E_{gh}\right) \leq \sum_g d = d|V(G)|.$$

Therefore, $rk\left(\sum_h E_{gh}\right) = d \quad \forall g \in V(G)$

$$\Rightarrow \quad \sum_h E_{gh} = I \quad \forall g \in V(G).$$

Example with Ω_n

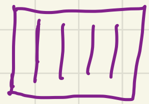
Recall: Ω_n is the orthagonality graph of $\{\pm 1\}^n$

$\chi_q(\Omega_{4n}) = 4n$ but $\chi(\Omega_{4n})$ is exponential in n.

Frankl + Rödl: $\alpha(\Omega_{4n}) \leq \gamma^{4n}$ for some $\gamma < 2$.

$\frac{|V(\Omega_{4n})|}{\alpha} \geq \left(\frac{2}{\gamma}\right)^{4n}$

Lemma: $\alpha(G \ltimes K_n) = \alpha(G \square K_n) \leq n\,\alpha(G)$



Corollary (Scarpa): $\alpha_q(\Omega_{4n} \square K_{4n}) = |V(\Omega_{4n})| = 2^{4n}$

but $\alpha(\Omega_{4n} \square K_{4n}) \leq 4n\,\gamma^{4n}$ for some $\gamma < 2$.

Thus $\dfrac{\alpha_q(\Omega_{4n} \square K_{4n})}{\alpha(\Omega_{4n} \square K_{4n})} \geq \dfrac{1}{4n}\left(\dfrac{2}{\gamma}\right)^{4n}.$

Another KS set

(0, 0, 0, 1)	(0, 0, 0, 1)	(1, −1, 1, −1)	(1, −1, 1, −1)	(0, 0, 1, 0)	(1, −1, −1, 1)	(1, 1, −1, 1)	(1, 1, −1, 1)	(1, 1, 1, −1)
(0, 0, 1, 0)	(0, 1, 0, 0)	(1, −1, −1, 1)	(1, 1, 1, 1)	(0, 1, 0, 0)	(1, 1, 1, 1)	(1, 1, 1, −1)	(−1, 1, 1, 1)	(−1, 1, 1, 1)
(1, 1, 0, 0)	(1, 0, 1, 0)	(1, 1, 0, 0)	(1, 0, −1, 0)	(1, 0, 0, 1)	(1, 0, 0, −1)	(1, −1, 0, 0)	(1, 0, 1, 0)	(1, 0, 0, 1)
(1, −1, 0, 0)	(1, 0, −1, 0)	(0, 0, 1, 1)	(0, 1, 0, −1)	(1, 0, 0, −1)	(0, 1, −1, 0)	(0, 0, 1, 1)	(0, 1, 0, −1)	(0, 1, −1, 0)

9 bases, each vector appears in 2 — Teresa Piovesan

We can think of each basis as a vertex + each vector as an edge.

This gives a graph G on 9 vertices with degree 4.

$$G \cong \text{Paley}(9) \cong K_3 \square K_3 \cong K_3 \times K_3 \cong L(K_{3,3})$$

$$|V(L(G))| = 18, \quad \omega(L(G)) = 4, \quad \chi_q(L(G)) = 4 < \frac{18}{4} \leq \chi(L(G)) = 5, \quad \xi_f(L(G)) = 4$$

$L(G)$ vtx transitive $\Rightarrow \chi_f(\overline{L(G)}) = \frac{|V(L(G))|}{\omega(L(G))} = \frac{18}{4} = 4.5, \quad \alpha_p(L(G)) = \frac{|V(L(G))|}{\xi_f(L(G))} = \frac{18}{4} \Rightarrow \alpha_q(L(G)) = 4 = \alpha(L(G))$

Theorem (R.): $\alpha_p(G) \doteq \chi_f(\overline{G}) \iff \exists r \in \mathbb{N}$ s.t. $\alpha_q(G[\overline{K_r}]) = r\chi_f(\overline{G})$

$\iff \exists r \in \mathbb{N}$ s.t. $\alpha_q(rG) = r\chi_f(\overline{G})$

$\alpha_q(2L(G)) = 9 > 2\alpha_q(L(G))$!!!

$$\chi_q \geq \xi_f \geq \frac{|V(L(G))|}{\alpha_p(L(G))}$$

$\alpha_q = \lfloor \alpha_p \rfloor$?

Lovász Theta

G a graph with adjacency matrix A.

Hoffman: $\chi(G) \geq 1 - \frac{\lambda_{max}(A)}{\lambda_{min}(A)} = \lambda_{max}\left(I + \frac{1}{-\lambda_{min}(A)} A\right)$

The only property of A needed is that

$A_{uv} = 0$ if $u \nsim v$ (includes $u = v$)

Optimizing Hoffman's bound over suitable A gives Lovász theta function of $\overline{G}$.

$$\vartheta(\overline{G}) = \overline{\vartheta}(G) = \max \lambda_{max}(A + I)$$
$$\text{s.t. } A_{uv} = 0 \text{ if } u \nsim v$$
$$\lambda_{min}(A) \geq -1 \quad A + I \succeq 0$$

$$= \min t$$
$$\text{s.t. } M_{uu} = t - 1 \quad \forall u \in V(G)$$
$$M_{uv} = -1 \quad \text{if } u \sim v$$
$$M \succeq 0$$

Max + min formulations mean that we have "certificates" for both $\overline{\vartheta}(G) \geq x$ + $\overline{\vartheta}(G) \leq y$.

Historical importance

On the Shannon capacity of a graph. Lovász, 1979

$$\alpha(G) \le \vartheta(G) \le \chi(\overline{G})$$

$$\vartheta(G \boxtimes H) = \vartheta(G)\vartheta(H) \Rightarrow \vartheta(G^{\boxtimes n}) = \vartheta(G)^n$$

$$\Rightarrow \Theta(G) := \lim_{n\to\infty} \left(\alpha(G^{\boxtimes n})\right)^{1/n} \le \vartheta(G)$$

Using this Lovász showed that

$$\Theta(C_5) = \sqrt{5}$$

$\Theta(C_{2k+1})$ for $k \ge 3$ is still open

Properties

$$\chi_f(G)\,\alpha(G) \ge |V(G)|$$

- $\vartheta(G)\overline{\vartheta}(G) \ge |V(G)|$ w/ equality if G is vtx trans.
- if G is vtx + edge transitive then $\overline{\vartheta}(G) = 1 - \frac{\lambda_{max}(A)}{\lambda_{min}(A)}$ for A the adjacency matrix of G.
- $\overline{\vartheta}(K_n) = n$ Exercise.
- "efficiently computable"
- many many formulations
- References: Lovász' paper (~ means adjacent or equal)
  The sandwich theorem by Knuth

Formulation in terms of projective packings

Recall that the value of a proj. pack. $g \mapsto E_g \in \mathbb{C}^{d \times d}$ is $\frac{1}{d}\sum_g rk(E_g) = \frac{1}{d}\sum_g Tr(E_g) = \frac{1}{d} Tr(\sum_g E_g)$

= avg of e-vals of $\sum_g E_g$

Proposition: $\vartheta(G) = \max \lambda_{max}(\sum_g E_g)$

s.t. $g \mapsto E_g$ is a proj. pack.

$\alpha_p(G) \leq \vartheta(G)$

More typical formulation: $\vartheta(G) = \max \sum_g |\langle \varphi | \psi_g \rangle|^2$

s.t. $g \mapsto |\psi_g\rangle$ ortho rep.
& $|\varphi\rangle$ unit vec.

Proof of equivalence: Exercise.

Monotonicity

Want to prove a "quantum sandwich theorem" i.e. that

$$\omega_q(G) \leq \bar{\vartheta}(G) \leq \chi_q(G).$$

Suffices to prove $G \xrightarrow{q} H \Rightarrow \bar{\vartheta}(G) \leq \bar{\vartheta}(H)$

let $m = \omega_q(G)$ $\quad n = \chi_q(G)$

$$K_m \xrightarrow{q} G \xrightarrow{q} K_n \Rightarrow \underbrace{\bar{\vartheta}(K_m)}_{m} \leq \bar{\vartheta}(G) \leq \underbrace{\bar{\vartheta}(K_n)}_{n}$$

Proof: Let $g \mapsto E_g \in \mathbb{C}^{d \times d}$ be a proj. pack. of $\bar{G}$ with $\lambda_{max}(\sum_g E_g) = \bar{\vartheta}(G)$. Suppose that $P_{gh} \in \mathbb{C}^{k \times k}$ for $g \in V(G)$ & $h \in V(H)$ give a quantum homomorphism from G to H. Define

$$F_h = \sum_g E_g \otimes P_{gh} \quad \forall\, h \in V(H).$$

F_h is a projection: each term $E_g \otimes P_{gh}$ is a projection &

$$(E_g \otimes P_{gh})(E_{g'} \otimes P_{g'h}) = \underbrace{E_g E_{g'}}_{=0 \text{ if } g \not\sim g' \,\&\, g \neq g'} \otimes \underbrace{P_{gh} P_{g'h}}_{=0 \text{ if } g \sim g'} = 0 \text{ if } g \neq g'$$

$h \mapsto F_h$ is a proj. pack. of $\overline{H}$: if $h \not\sim h'$ & $h \neq h'$ then

$$F_h F_{h'} = \sum_{g,g'} \underbrace{E_g E_{g'}}_{\substack{=0 \text{ if } g \not\sim g' \\ \& \ g \neq g'}} \otimes \underbrace{P_{gh} P_{g'h'}}_{\substack{=0 \text{ if } g = g' \\ \text{or } g \sim g'}} = 0$$

Lastly, $\sum_h F_h = \sum_{g,h} E_g \otimes P_{gh} = \sum_g E_g \otimes \left(\sum_h P_{gh}\right) = \left(\sum_g E_g\right) \otimes I$

$$\Rightarrow \overline{\vartheta}(H) \geq \lambda_{max}\left(\sum_h F_h\right) = \lambda_{max}\left(\sum_g E_g\right) = \overline{\vartheta}(G). \quad \square$$

Application: Ω_{4n} is vtx & edge transitive

$$\overline{\vartheta}(\Omega_{4n}) = 4n \qquad \vartheta(\Omega_{4n}) = \frac{2^{4n}}{4n}$$

So $\chi_q(\Omega_{4n}) \geq 4n$ & thus $\chi_q(\Omega_{4n}) = 4n$.

Also $\alpha_q(\Omega_{4n}) \leq \left\lfloor \frac{2^{4n}}{4n} \right\rfloor$. If n is not a power of 2, then

$$\frac{|V(\Omega_{4n})|}{\alpha_q(\Omega_{4n})} > \frac{2^{4n}}{(2^{4n}/4n)} = 4n = \chi_q(\Omega_{4n}) \ !!!$$